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Renormalizable extra-dimensional models

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Abstract: Non-abelian gauge theories may have continuum limits in more than four dimensions, supported by non-trivial ultra-violet fixed points. Moreover, such theories can be expected to be accessible to Wilson's epsilon expansion. We investigate this series for SU(N) Yang-Mills, in particular for the fixed point coupling and exponent ν, up to four loops. From the model-building point of view, such theories would be effectively perturbatively renormalizable in the normal way. A particularly attractive possibility is the construction of renormalizable extra-dimensional models of the weak interactions, which have the potential to address the full hierarchy problem. The simplest such gauge-Higgs unification model is however ruled out by a combination of theoretical and phenomenological constraints.

Keywords: Field Theories in Higher Dimensions, Nonperturbative Effects, Beyond Standard Model, Compactification and String Models.

Contents

1. Introduction

There has been considerable interest in the last few years in constructing field theories of particle physics in more than four dimensions. On the one hand, these ideas are inspired by similar features in string theory, on the other hand, elegant new approaches to the persistent mysteries of theoretical particle physics are now possible [1, 2]. These aim variously to solve the famous hierarchy problem (i.e. the fact that the Higgs mass is quadratically sensitive to higher scales), the little hierarchy problem (the fact that precision measurements seem to favour a scenario with a low Higgs mass $\sim 115\,\mathrm{GeV}$ and no new physics until $\sim 10\,\mathrm{TeV}$ [3]), the pattern of fermion masses and mixing angles, and problems with grand unified theories such as doublet-triplet splitting, proton lifetime and mass relations. The new approaches to solving these issues include theories with so-called universal extra dimensions which e.g. result in calculable SUSY breaking and Higgs masses [4], gauge-Higgs unification in which the Higgs field is a component of a higher dimensional gauge field whose mass is thus protected by gauge invariance [5]–[13], Higgsless models where the symmetry is broken by the boundary conditions [14], grand unified models in which colour triplet Higgs are given a Kaluza-Klein mass whilst the doublet remains massless at the compactification scale [15] (see also [16]), and approaches to the flavour problem using the freedom to place fermions at different points (branes) in the extra dimensions and to couple them non-locally through other fields or Wilson lines [17, 18, 19, 20, 9, 13].

However all of these new approaches suffer from a severe drawback: quantum field theory as conventionally envisaged is not renormalizable in more than four space-time dimensions. Thus these theories must be viewed as effective field theories, only applicable over a limited energy range. For energies much lower than the defining scale E in the theory (e.g. $E \sim 1/R$ where R is the compactification scale) experimental constraints require that it reduce to the Standard Model plus very small corrections. On the other hand there is a maximum energy Λ, usually taken to be the energy where some coupling becomes strong and estimated by so-called naive dimensional analysis (NDA) [21], above which something other than quantum field theory must take over (for example some conjectural string theory). This maximum energy depends on the couplings, with larger Λ/E for smaller couplings, but typically $\Lambda \sim 10E$ to $100E$ is possible. With the ultraviolet completion unknown, we can only bound the symmetry-preserving higher dimensional vertices in the effective theory, under the assumption that they naturally have order one coefficients at the cutoff scale. This sets an irreducible limit on the expected predictivity of these models controlled by E/Λ, again dependent on the details, but typically authors aim for $\sim 1\%$.

This limitation is less of a problem for theories that attempt to describe the energy range between the GUT scale 10^{15} GeV and the Planck mass 10^{19} GeV, and for which the irreducible uncertainty of $\sim 1\%$ comes on top of already (better be) small corrections to Standard Model quantities.

It is a much more drastic issue for higher dimensional theories that directly address the hierarchy problem. These models tackle the problem by introducing new physics at $E \sim 1\,\text{TeV}$ in such a way that the Higgs mass becomes insensitive to higher scales. Here attempts to bring such a model into agreement with the impressive precision LEPI/II data can in turn force it into regions of parameter space where the need for large ratios in couplings then result by NDA in significant reductions in Λ, cutting off its domain of applicability and thus severely limiting its predictability (e.g. see [13, 22]). Since an as yet unknown ultraviolet completion at some intermediate scale Λ is required, these models can actually only address the little hierarchy problem, because any insensitivity of the Higgs mass to scales higher than Λ has to be explained in the unknown theory. It is in any case a rather drastic step to address the little hierarchy problem by abandoning quantum field theory altogether at say $\sim 100\,\text{TeV}$ for a largely conjectural ultraviolet non-field theory.

It might be possible however, to rest certain restricted classes of such models on a more secure foundation. In the 1970s it seems to have been commonly understood that non-abelian Yang-Mills theory may exist as a non-perturbative quantum field theory, a.k.a. continuum limit, in more than four space-time dimensions [23]. With few exceptions since then [24]–[31] this possibility seems to have been ignored or forgotten. Independently in string theory, probably related supersymmetric versions have been discovered [32].[1]

We briefly review the little evidence that has collected, for and against such theories, in section 2. We also investigate the series for the fixed point coupling $\tilde{\alpha}_*$, and the critical exponent ν, using the ϵ expansion up to four loops. Together, the evidence suggests that these fixed points exist in $D = 5$ dimensions, possibly in $D = 6$ dimensions and probably not for $D \geq 7$ dimensions.

At any rate such theories formally do exist within an expansion in ϵ of a $4 + \epsilon$ dimensional theory. As we review in section 2, this means that quantum corrections can be computed perturbatively and renormalized in *four* dimensions. From the model building point of view, the theory is perturbatively renormalizable; one only needs to bear in mind that at high energies the appropriate dimensionless ratio $\tilde{\alpha}$, involving the Yang-Mills coupling α, runs to a computable finite value rather than zero.

[1] A Kaluza-Klein treatment appears in ref. [33].

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For the continuum limit to exist within the ϵ expansion when other fields are included, one requires that in four dimensions, all couplings are relevant (effectively masses and couplings of positive mass dimension) or marginally relevant (equivalently asymptotically free). We outline the restrictions this imposes in section 3. We then extend this understanding of renormalizability to models with branes, and discuss the constraints that arise, concentrating on the bosonic sector, as well as making some further comments on the applicability of this idea to Grand Unified Theories.

A particularly attractive application is to extra dimensional models of the weak interactions: in this case the insensitivity of the Higgs mass to scales up to Λ, become solutions to the hierarchy problem, since by the definition of renormalizability, Λ can be taken to infinity. We explore this in section 3 and section 4. We are led to consider models of the weak interactions based on the Hosotani mechanism [34, 35, 36].

We show however in section 4, where the Hosotani mechanism, and the fermionic sector are considered in detail, that the simplest such model, based on a bulk SU(3) gauge theory, is ruled out. In particular we show that in order to get the right Yukawa interactions, we are forced to add sufficiently many bulk matter fields that the SU(3) Yang-Mills is no longer asymptotically free in four dimensions and thus has no continuum limit within the ϵ expansion of $4+\epsilon$ dimensions. As we note in our conclusions (section 5) this only means that one should try to construct more involved models, for example with a larger gauge group, and in addition introduce the bulk fields more economically.

We start in section 2 with a short explanation of the ϵ expansion, comparing to its use in its more traditional context. We then review the evidence such as it is for higher dimensional fixed points. After this we use the known four-loop four-dimensional beta function [37, 38] to investigate the ϵ expansion for these fixed points. (It seems that this is the first time the early two-loop investigation [23] has been extended.)

2. Renormalizability in extra dimensions

Scalar field theory in three dimensions has a Wilson-Fisher fixed point [39]. In the case of an N-component scalar field theory with $O(N)$ invariance, tuning the couplings so that the theory lies close to this fixed point results in universal behaviour in a universality class determined only by N (see e.g. ref. [40]). The 'distance' from the fixed point sets a mass-scale m (the inverse correlation length) which is in fact the only parameter left in the universal physics.[2] Exactly at the fixed point, one obtains an interacting conformal field theory. The properties of the conformal field theory determine everything that is universal about this fixed point, for example the scaling dimensions of its operators determine the various critical exponents ν, ω, η etc., pure numbers that are amongst the simplest quantities to calculate and/or measure, while the effective potential for ϕ yields the universal equation of state. Viewed in this way, this conformal field theory and its leading relevant deformation constitute a concrete example of non-trivial quantum field theory, i.e. a non-trivial continuum limit, about a fixed point other than the gaussian fixed point.

[2] There is one relevant coupling, which can be parametrized by m, and there are no marginal couplings.

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As we will see, in a closely analogous way, non-trivial continuum limits may well exist for Yang-Mills theory in higher than four dimensions. In order to gain a complete appreciation of these phenomenona one needs to use the language of the Wilsonian renormalization group [39]. For clarity we remind the reader of the basic elements. One works within an infinite dimensional space of bare actions that include all possible local interactions allowed by the symmetries of the theory. In this space, there is the so-called critical manifold, which consists of all bare actions yielding a given conformally invariant continuum limit (for example the $O(N)$ conformal field theories above or the non-perturbative gauge theory cases we are about to discuss). Any point on this manifold — i.e. any such bare action — flows under a given Wilsonian renormalization group towards its fixed point; local to the fixed point, the critical manifold is spanned by an infinite[3] set of irrelevant operators. The other directions emanating out of the critical manifold at the fixed point, are spanned by relevant and marginally relevant perturbations (couplings with positive and vanishing scaling dimensions respectively, but where the latter grow as we move to lower energies, i.e. are asymptotically free). Next, we choose a bare action with sufficiently many operators of the right type to intersect the critical surface, for some choices of couplings. Note that in general we have no way of knowing what type of bare action will do: for a non-perturbative fixed point we cannot rely on naïve dimensional analysis to classify which operators are (marginally) relevant. Instead we simply need to search the full space.

Now in the bare action, we shift a coupling a little bit away from the critical manifold. The trajectory of the Wilsonian renormalization group will to begin with, move towards the fixed point, but then shoot away along one of the relevant directions towards the so-called high temperature fixed point which represents a theory with only infinite mass scales.

To obtain the continuum limit, and thus a finite mass scale, one must then tune the bare action towards the critical manifold and at the same time, reexpress physical quantities in renormalized terms appropriate for the diverging correlation length.

To *confirm* the existence of such a continuum limit, one really has no choice but to follow through the above procedure for example within the framework of a lattice gauge theory computation in higher than four dimensions.

Unfortunately, to date all such investigations have been limited, with ambiguous conclusions. Thus in ref. [24], an exact renormalization group treatment was used, which taken at face value implies that the non-trivial fixed point exists for SU(N) Yang-Mills in $D = 5$ dimensions for $N \leq 5$. The author computed the critical dimension D_{cr} above which the fixed point disappears, finding $5 < D_{cr} < 6$ in all these cases. Actually, in ref. [24] $N = 4$ is not reported but the results for $N = 2, 3, 5$ do suggest that the critical dimension D_{cr} behaves smoothly with N as expected. The problem with such a treatment is primarily that there is no control of uncertainties due to truncation of the flow equations. There are a small number of lattice studies [25]–[28]. These are subject of course to limitations from systematic and statistical errors, which rapidly become more severe with increasing dimension (and increasing N). The general conclusion seems to be that a non-trivial continuum

[3]This is strictly an assumption on our part for the gauge theory cases of interest.

limit does not exist in $D > 4$ dimensions for the simple Wilson plaquette action (at least for $N = 2$ and $D = 5, 6$). However as we have already emphasised, there is no reason to expect that this simplest action is the correct one in this case.[4] Ref. [28] displayed evidence for a non-trivial continuum limit even with the simple Wilson plaquette action, providing the extra dimensions are small enough, but it is reasonable to assume that this arises from lattice artifacts [31, 24]. Ref. [26] reported on the addition of an adjoint representation Wilson plaquette but the results were inconclusive. Only in ref. [27] were negative conclusions drawn from more general actions than the simple Wilson plaquette action, but this was in the special case of $D = 6$ dimensions with large N (=27 and 64) after reduction to a twisted Eguchi-Kawai model.

If these non-trivial fixed points exist, one of the most powerful ways of deriving the physics resulting from these fixed point behaviours is by computing perturbative quantum field theory in four dimensions and using the so-called ϵ expansion [40].

The idea is as follows. In four dimensions we know that the perturbative Yang-Mills theory is based around the gaussian fixed point, and parametrized by a small marginally relevant coupling g (just as perturbative scalar field theory is parametrized by a small marginally irrelevant coupling λ and a relevant mass term). All the independent higher point interactions (field strength cubed etc., in the scalar case ϕ^6 etc.) are irrelevant; their effect just amounts to a finite renormalization of the low energy couplings. Because the theory is free at the associated fixed point, the scaling dimensions of the irrelevant operators are equal to their engineering dimensions, and thus are 6 or greater. Now, providing that the scaling dimensions of these operators are continuous functions of the dimension D, it must be that they are still irrelevant in $D = 4 + \epsilon$ dimensions, for small enough ϵ. Thus for small enough ϵ, the usual bare action and renormalized effective lagrangian must still yield the right description.

In the scalar case, this results in the classic argument for understanding the existence of the Wilson-Fisher fixed point: the beta function takes the form

$$\mu\partial_\mu\tilde{\lambda} = \epsilon\tilde{\lambda} + \beta(\tilde{\lambda})\,, \tag{2.1}$$

where the first term is classical and arises simply because we use the Wilsonian coupling, namely the dimensionless combination $\tilde{\lambda} = \lambda\mu^\epsilon$, while the second — quantum — term can coincide with the beta function in four dimensions (as we will see explicitly later). Since below four dimensions the classical term is negative while the quantum term starts at higher powers than $\tilde{\lambda}$ and is positive (reflecting triviality of four dimensional scalar field theory), we get a zero at some point $\tilde{\lambda} = \tilde{\lambda}_*$, at least for sufficiently small ϵ, corresponding to an infrared attractive fixed point for $\tilde{\lambda}$.

In the gauge theory case, we have qualitatively the same situation but with the signs reversed. The quantum term can be chosen to be the four-dimensional beta function and is now negative (corresponding to asymptotic freedom) and this can be balanced by going above four dimensions, where the classical term is positive (corresponding to a negative dimension coupling g). Again, we get a fixed point $\tilde{g} = \tilde{g}_*$, for sufficiently small ϵ, but this

[4]See also ref. [24]. This is independent of a possible multicriticality of the fixed point however.

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time it corresponds to the desired ultraviolet attractive fixed point. Indeed we can now compute this fixed point and its universal consequences by the ϵ expansion, i.e. simply by solving for $\tilde{g}_*$ order by order in ϵ.

In fact for the gauge theory, ϵ expansion of non-linear sigma models is even more closely analogous [40, 41, 42]. $O(N)$ invariant non-linear sigma models in two dimensions are also renormalizable and asymptotically free. By the same logic, we expect that in $D = 2 + \epsilon$ dimensions, there is still a continuum limit for these sigma models but now based around a non-trivial ultraviolet attractive fixed point. This fixed point indeed exists and is nothing but the infrared description of the $O(N)$ Wilson-Fisher fixed points already discussed (as may be confirmed for example by noting that the large-N expansion of this fixed point coincides with the large N expansion of the one obtained in D dimensional scalar field theory) [40].

(There is some controversy over the reasons for the poor accuracy in practice of the ϵ expansion for these non-linear sigma models [41, 42]; some crucial non-perturbative effect is perhaps missing. At any rate the qualitative conclusion is correct, namely that ultraviolet fixed points exist above the upper-critical dimension where these theories cease to be perturbatively renormalizable. It remains to be understood whether the issues of accuracy have any bearing on the case of Yang-Mills we discuss here.)

With respect to the use of the ϵ expansion in all these examples, a word of caution is in order about taking the description too literally. Although the ϵ expansion can furnish impressively accurate answers for universal quantities [40] (the renormalized quantities, which are all we are interested in), of course the four dimensional field theory has very different properties in general from the $D \neq 4$ theory (which latter needs to be understood from a Wilsonian renormalization group perspective, as already emphasised). Thus, for example for the $O(N)$ scalar field theory in three dimensions, perturbation theory about the gaussian fixed point generically includes the now marginal ϕ^6 interaction. More importantly, perturbative three dimensional ϕ^4 field theory is superrenormalizable. Its ultraviolet divergences do not give rise to the four dimensional β function in (2.1). Indeed λ suffers no ultraviolet divergences at all; only m^2 receives divergent corrections at one and two loops. Instead, its massless limit is plagued with infrared divergences, a signal that we are working about the wrong fixed point. In $4 + \epsilon$ dimensions, with $-1 < \epsilon < 0$, even for arbitrarily small ϵ, ϕ^4 scalar field theory is still superrenormalizable and its massless limit suffers infrared divergences, first arising at $\sim -2/\epsilon$ loops (as follows simply from power counting) [40]. In gauge theory in five dimensions we cannot *a priori* exclude any of the full infinity of higher dimensional gauge invariant operators from a valid bare action. The theory in five dimensions is perturbatively non-renormalizable; it is plagued with infinitely many new types of ultraviolet divergence. In particular, its ultraviolet divergences do not give rise to the four dimensional β function in (2.4). Even in $4 + \epsilon$ dimensions, with arbitrarily small $\epsilon > 0$, Yang-Mills theory has non-renormalizable divergences that appear first at $\sim 2/\epsilon$ loops (again simply by power counting).

However, the ϵ expansion allows us to exchange the limits ($\epsilon \to 0$ and $g \to 0$) and follow perturbatively the relevant fixed point, and thus in this way access the required continuum limit (i.e. the renormalized quantities) directly.

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We now turn to the details. We write the gauge field in D dimensions as $\mathcal{A}_M = \mathcal{A}^a_M T^a$, where the T^a are generators of some simple gauge group G, with standard normalisation $\mathrm{tr}\ T^aT^b = \delta^{ab}/2$. (We use a metric of signature $+-- \cdots$.) Writing the covariant derivative as $\nabla_M = \partial_M - i\mathcal{A}_M$, the field strength is $\mathcal{F}_{MN} = i[\nabla_M, \nabla_N]$. In $D = 4$ dimensions the lagrangian density takes the usual form

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{tr}\,\mathcal{F}^{MN}\mathcal{F}_{MN}\,. \tag{2.2}$$

As we have already stated, we expect that the situation is continuous in ϵ and thus in $D = 4+\epsilon$ dimensions, (2.2) is still the right description for small enough ϵ. At this point it is helpful to recall a textbook argument. In dimensional regularisation and minimal subtraction, we replace the coupling in (2.2) by g_0, and use the dimensionless renormalized $\tilde{g} = \mu^{\epsilon/2} g$ to derive

$$g_0 = \mu^{-\epsilon/2}\left(\tilde{g} + \sum_1^\infty \frac{Z_n(\tilde{g})}{\epsilon^n}\right), \tag{2.3}$$

where the $Z_n(\tilde{g}) = O(\tilde{g}^{2n+1})$ are the usual power series in $\tilde{g}$ with numerical coefficients chosen to cancel all the poles that arise order by order in perturbation theory as $\epsilon \to 0$. Now we differentiate (2.3) with respect to μ. Since the theory is finite when expressed in terms of $\tilde{g}$, and using the fact that g_0 does not depend on μ, we can invert the result to obtain

$$\mu\partial_\mu \tilde{g} = \frac{\epsilon}{2}\tilde{g} + \beta(\tilde{g})\,, \tag{2.4}$$

where $\beta(\tilde{g}) = Z_1(\tilde{g})/2 - \tilde{g}Z_1'(\tilde{g})/2$, and all the Z_i with $i > 1$ are determined in terms of Z_1 by the requirement that all the $1/\epsilon$ poles cancel in (2.4).

As promised, we see that in a little more than four dimensions ($\epsilon > 0$) we can balance the first positive term against the negative power series $\beta(\tilde{g})$ to get a fixed point $\tilde{g}_*$, whose properties we can now compute order by order in ϵ.

For what ensues, it is especially important to note that even though (2.4) expresses the flow of $\tilde{g}$ in more than four dimensions, $\beta(\tilde{g})$ is *precisely the usual four dimensional beta function.* Thus to deduce the properties of the model in greater than four dimensions, we only need renormalize the model in the normal way in four dimensions and then use (2.4) and similar equations as a formal device to analytically continue to $\epsilon > 0$.

As we will now show, the ϵ expansion for these ultraviolet fixed points is in fact surprisingly well behaved. We can illustrate this using the known four-loop β function [37, 38] for SU(N) Yang-Mills.

At this point we note a convenient point of detail: the four-dimensional β function is the same in the MS and $\overline{\mathrm{MS}}$ schemes [38]. This can be understood from the definition of $\overline{\mathrm{MS}}$ namely that, instead of subtracting via counterterms, powers of $2\tilde{g}^2/(4\pi)^2$ times poles in ϵ, one substracts powers of

$$\frac{2\tilde{g}^2}{(4\pi)^2}\{1 + \ln(4\pi)\epsilon - \gamma_E\epsilon\}$$

(γ_E being Euler's constant), the corrections arising from

$$\Omega_D = \frac{2}{(4\pi)^{D/2}\Gamma(D/2)}\,,$$

the D dimensional solid angle divided by $(2\pi)^D$ that accompanies each loop integral. Therefore for the Z_n in (2.3), $\overline{\mathrm{MS}}$ amounts to an ϵ dependent rescaling of $\tilde{g}$. When working at finite ϵ, the results can be expected to be better behaved if we instead absorb the full dependence coming from Ω_D [40]. This amounts to replacing $2\tilde{g}^2/(4\pi)^2$ in the MS scheme, with $\tilde{g}^2\Omega_D$. Clearly this modified MS also yields the same four-dimensional $\beta(\tilde{g})$. Finally for convenience in the ensuing analysis, we also absorb a factor of N and write

$$\tilde{\alpha} := N\Omega_D\,\tilde{g}^2 = N\Omega_D\,\mu^\epsilon g^2 \tag{2.5}$$

In terms of $\tilde{\alpha}$, (2.4) becomes

$$\mu\partial_\mu\tilde{\alpha} = \epsilon\tilde{\alpha} + \beta(\tilde{\alpha})\,, \tag{2.6}$$

where

$$\beta(\tilde{\alpha}) = -\beta_0\tilde{\alpha}^2 - \beta_1\tilde{\alpha}^3 - \beta_2\tilde{\alpha}^4 - \beta_3\tilde{\alpha}^5 + O(\tilde{\alpha}^6)\,; \tag{2.7}$$

these β_n are the coefficients quoted in ref. [38] but divided by 2^nN^{n+1}.

We compute $\tilde{\alpha}_*$, although this is not universal, but we also compute the index ν via [23]

$$\frac{1}{\nu} := -\frac{d\beta}{d\tilde{g}}(\tilde{g}_*) = -\frac{d\beta}{d\tilde{\alpha}}(\tilde{\alpha}_*)\,, \tag{2.8}$$

which is the universal power relating the analogous QCD-like scale Λ_{YM} to the distance from the fixed point at any high scale μ where the coupling is $\tilde{\alpha}(\mu)$ is sufficiently close to $\tilde{\alpha}_*$:

$$\Lambda_{YM} \propto \mu|\tilde{\alpha} - \tilde{\alpha}_*|^\nu\,. \tag{2.9}$$

Solving (2.6) for $\tilde{\alpha}_*$,

$$\tilde{\alpha}_* = \frac{\epsilon}{\beta_0} - \frac{\beta_1\epsilon^2}{\beta_0^3} + \left(2\frac{\beta_1^2}{\beta_0^5} - \frac{\beta_2}{\beta_0^4}\right)\epsilon^3 + \left(5\frac{\beta_2\beta_1}{\beta_0^6} - \frac{\beta_3}{\beta_0^5} - 5\frac{\beta_1^3}{\beta_0^7}\right)\epsilon^4 + O(\epsilon^5)\,, \tag{2.10}$$

and substituting the values for β_n, we find a surprise. Whereas the coefficients β_n show the expected dramatic increase in n, consistent with (2.6) being asymptotic (even at $N = \infty$):

$$\beta_0 = 3.667\,, \qquad \beta_1 = 5.667\,, \qquad \beta_2 = 13.23\,, \qquad \beta_3 = 39.43 + \frac{51.22}{N^2}\,, \tag{2.11}$$

the series for $\tilde{\alpha}_*$

$$\tilde{\alpha}_* = .2727\epsilon - .1150\epsilon^2 + .02372\epsilon^3 - \left(.007395 + \frac{.07729}{N^2}\right)\epsilon^4\,, \tag{2.12}$$

is considerably better behaved, and at $N = \infty$ it is not at first clear that this series is asymptotic. Substituting (2.10) in (2.8), we obtain

$$\frac{1}{\nu} = \epsilon + .4215\epsilon^2 + .1813\epsilon^3 + \left(.1242 + \frac{.8502}{N^2}\right)\epsilon^4\,. \tag{2.13}$$

Again, at $N = \infty$, this series looks very well behaved.

We start by investigating this limit in more detail. Writing the coefficients of ϵ^n in (2.12) and (2.13) as a_n and ν_n respectively, successive ratios $|a_n/a_{n+1}|$ give $2.37, 4.84, 3.21$ for $n = 1, 2, 3$. And similarly successive ratios $|\nu_n/\nu_{n+1}|$ give $2.37, 2.32, 1.45$. If we did not know better, we might even be tempted to conclude that these ratios have a limit $r > 0$, corresponding to a finite radius of convergence r. However, we see from (2.12), that $\tilde{\alpha}_*$ has negative values for all $\epsilon < 0$, as expected, since the four dimensional beta function analytically continued to $\alpha < 0$ behaves like that of a trivial theory and thus allows us to balance the two terms in (2.6) and find a fixed point in $D < 4$ dimensions. Since the theory is non-perturbatively sick for $\alpha < 0$ [43], the above series in ϵ must have zero radius of convergence.

If the series are already displaying their limiting asymptotic behaviour (viz. $a_n \sim cs^n\Gamma(n+\zeta)/R^n$, for $s = -1$ and some constants c, ζ and R, and similarly for ν_n, with $s = 1$)[5] then we can estimate them by taking the sum to the point where the terms stop getting smaller (including the smallest term) and using the next term to estimate the error. In this case we ought to see $|a_n/a_{n+1}| \sim R/(n+\zeta)$, and thus $\lim_{n\to\infty} n|a_n/a_{n+1}| = R$. Forming this combination for $n = 1, 2, 3$, we get for the a_ns: $2.37, 9.69, 9.62$, and for the ν_ns: $2.37, 4.65, 4.38$.

We can already draw a number of important conclusions. Firstly, there is clear evidence that the series do approximate their limiting behaviour already after the first term. Secondly, the fact that the last two combinations are so close to each other, for both series, indicates that the corresponding ζ must be small in these cases; of course the series are not long enough to reliably estimate it however. Thirdly, we expect the $\sim R^{\text{th}}$ coefficient to be the smallest one, thus for $1/\nu$ in (2.13), we expect the next (5-loop) term to have a larger coefficient than the four-loop term, while for $\tilde{\alpha}_*$, we expect to have to go to 10 loops in (2.12) to see the coefficients start growing with n.

Fourthly, we can use the series to estimate ν and $\tilde{\alpha}_*$ for positive integer ϵ, as follows. For ν we simply estimate it as described above, from the asymptotic series for $1/\nu$; for $\epsilon = 1$ we have a special case in that we are missing the 5-loop term that would provide an estimate of the error. This is why there is no error for the appropriate entry in table 2.

$D\backslash N$	2	3	4	∞
5	.62(13)	.62(9)		.579
6	.19(20)	.19(13)	.19(11)	.19(8)

Table 1: Estimates for ν for varying N and D.

For $\tilde{\alpha}_*$ at $\epsilon = 1$ we have an alternating series in which we expect the terms to keep getting smaller until the $(R-1)^{\text{th}}$ term with the slightly larger R^{th} term providing the error, where $R \approx 10$. Clearly in this case $\tilde{\alpha}_*$ should lie between the sum of the first three and the sum of the first four terms.[6] This gives us the $D = 5$, $N = \infty$ estimate in table 2. In general the point where the terms stop getting smaller can be expected to be at $n = R/\epsilon$. Comparing with (2.12), we see that for integer $\epsilon > 1$ we should use the above method for summing asymptotic series, except that for $\epsilon = 2$ and 3, we are missing the term that would provide the error. This is why we do not display an error when $D = 6$ and $N = \infty$.

[5] Equivalently $a_n \sim c(-)^n n^\zeta n!/R^n$.

[6] If the coefficients kept getting smaller this would be a theorem.

Finally, at finite N, the series are clearly asymptotic, the $1/N^2$ terms in (2.12), (2.13) (the result of the first divergent non-planar contribution — which appears at four loops [38]) having a much larger coefficient (and of the same sign). Without further information (see below), we can estimate the sums only in the cases where this last term is not the smallest term. We find that the last term is smallest in (2.13) only when $D = 5$ and $N \geq 4$, and in (2.12) when $D = 5$ and $N \geq 3$, and when $D = 6$ and $N \geq 5$. This is why we the leave the corresponding entries blank in the tables.

$D\backslash N$	2	3	4	5	∞
5	.18(3)				.178(4)
6	.28(43)	.28(26)	.28(20)		.157

Table 2: Estimates for $\tilde{\alpha}_*$ for varying N and D.

In addition, we find that for $1/\nu$, the terms in the series only increase once $D \geq 7$ (for any N). For $\tilde{\alpha}_*$ when $D = 7$, the error is larger than the estimate for $2 \leq N \leq 12$, after which the last term is the smallest so that we cannot provide an estimate. At $N = \infty$, we saw above that we cannot estimate the error, and the series sums to $\tilde{\alpha}_* = -0.175$, which cannot make sense physically. For $D = 8$, the error is much larger than the estimate for $\tilde{\alpha}_*$ for all N. For $D \geq 9$, the terms only increase in magnitude with n. We take all this as evidence from the ϵ expansion that the fixed points do not exist in $D \geq 7$.

In $D = 6$ the evidence is somewhat marginal. At $N = 2$ the error is larger than the estimate, both for ν and $\tilde{\alpha}_*$. Perhaps this indicates that the fixed point does not exist. For ν the errors are large but gradually decreasing for $N \geq 3$ and N increasing, but even at $N = \infty$ the error is 40%. For $\tilde{\alpha}_*$ the errors remain large where they can be estimated at all.

On the other hand, for $D = 5$ we appear to have clear evidence for a fixed point: the errors are small where they can be estimated. There are missing entries only because the series are too short in these cases. Given the other values it is reasonable to assume that $\tilde{\alpha}_* \sim .18$ and $\nu \sim .6$ for all $N \geq 2$.

These conclusions are in broad agreement with the earlier alternative approaches already discussed [24]–[28], given the limitations with all these studies.

We have only performed the simplest estimates. It should be possible to do much better. In the lower dimensional scalar field theories, powerful techniques have been developed to cope with their divergent ϵ expansions. One first transforms the series to the Borel plane. Relying on no other knowledge the series can be resummed by Padé approximants. However, if one assumes the ϵ expansion inherits the large order behaviour of $\lambda\phi^4$ theory in three dimensions one knows about the closest singularity in the Borel plane (from studying Lipatov instantons) [40]. By conformal mapping the Borel plane, this information can be taken into account. The results are impressive: for better known quantities e.g. $\tilde{\lambda}_*$ and ν (and using also information from the exactly soluble two-dimensional Ising model) the theoretical error has been reduced to a few per mille [40].

In the gauge theory case, one can also transform the series to the Borel plane and use Padé approximants. However, one also has similar knowledge about the large order behaviour of perturbation theory coming from the closest singularities in the Borel plane, in this case due to infrared renormalons and instantons, which can also be taken into account.

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3. Renormalizability with matter and branes

We pursue theories which have non-perturbative continuum limits in more than four spacetime dimensions, in the sense of having ultraviolet fixed points within the ϵ expansion. We have seen that this requires that when $\epsilon = 0$, the model must be renormalizable and in addition all couplings in the bulk are relevant (effectively masses and couplings of positive mass dimension) or marginally relevant (equivalently asymptotically free).[7]

Thus we can readily incorporate non-abelian gauge fields in the bulk of the higher dimensions. We can then add fermions to the bulk of the higher dimensions, providing there are not too many or that their representations are too large, such as would destroy the coupling's asymptotic freedom. It is harder to include scalars in the bulk (because the scalar self-coupling must also be asymptotically free), but possible for a careful choice of couplings. On the other hand, we cannot at the same time as keeping asymptotic freedom, use the scalars to give a mass to all the gauge fields via the Higgs mechanism [44]. Inclusion of both fermions and scalars allows asymptotic freedom only for a fine tuning of the Yukawa couplings which is natural only in supersymmetric theories [44]. Finally, abelian gauge fields can be included only on four dimensional branes if at all.

As is already clear, these models are highly restricted. We will however meet yet other constraints renormalizability puts on the couplings. We will concentrate on the case of five dimensional theories, i.e. where one should set $\epsilon = 1$ eventually. This is because, as we have seen, the evidence for non-trivial fixed points is strongest in five dimensions. However, most of our comments obviously extend to other dimensions. Our conventions are to write five dimensional indices in Latin capitals with components $M = 0, 1, 2, 3, 5$; coordinates we will write as $x^M = (x^\mu, y)$, while we write momenta $p_M = (\bar{p}_\mu, p_5)$, the bar being an extra flag for four-dimensional parts. We write the lagrangian density of the non-abelian gauge field as in eq. (2.2).

We do not consider gravity since we do not know how to describe it as a renormalizable theory. We thus take the conservative view that there is a flat classical gravitational background, to be set and not questioned. Phenomenologically therefore, these models have to have the fifth dimension compactified, which we can take to be a circle radius R: y is identified with $y + 2\pi R$. Although the models we consider, are valid as quantum field theories up to infinite energies, in practice we have to set the cutoff at the natural five-dimensional Planck mass since quantum gravitational effects cannot be ignored above this. (This means that $\Lambda \sim a^{1/3} \times 10^{13}\,\text{GeV}$, where we have taken the radius of compactification to be $1/R = a\,\text{TeV}$ [45].)

At first sight it is attractive when computing such a model in the ϵ expansion, to regard the extra ϵ dimensions as the ones that are compactified. This is not consistent however. In the ϵ expansion the extra dimensions only enter by the change in the scaling of the classical coupling constant. The calculation is thus not sensitive to topological or geometric features of the ϵ extra dimensions. This is reasonable: the approach to the

[7]Note that these are the same conditions that are believed to be required to ensure the non-perturbative existence of the quantum field theory in four dimensions.

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fixed point $\tilde{g}_*$, is a property of the theory in the far ultraviolet, i.e. at very small length scales, where space-time should look flat. Where a more precise matching is required between renormalized couplings above and below a compactification scale, the result from ϵ expansion with one of the *original* dimensions compactified, should clearly be used to compute threshold corrections.

If parts of the five dimensional gauge fields $\mathcal{A}_M$ are to represent some Standard Model gauge fields, then we need to go beyond trivial compactifications. (We do not observe the adjoint massless scalars $\mathcal{A}_5$ that would arise from such compactified Yang-Mills fields, and we cannot use extra-dimensional Higgs fields to give masses to all parts of $\mathcal{A}_5$.) There are two overlapping possibilities that have been suggested in the literature, the extra dimensional space can be an orbifold or more generally can have a boundary on which non-trivial boundary conditions are imposed [14].

Either way, once we take the model seriously as describing a renormalizable continuum limit, we have to introduce boundaries, a.k.a. branes, and boundary couplings which we cannot ignore, as we now explain.

In the general case [14], boundary conditions on the gauge fields are imposed that explicitly break the gauge group G down to some smaller (e.g. Standard Model) group H. Of course an explicitly broken gauge theory is not renormalizable. This is the only problem, and is not in fact a problem *per se*, for the little hierarchy solution envisaged by these authors, as can be understood by viewing the boundary conditions as imposed by boundary Higgs fields in the non-linear sigma model limit [46]. The effective cutoff for the sigma model can be identified with the effective cutoff the theory already necessarily has. However these problems do rule out general breaking by boundary conditions as an option here, because the non-linear sigma model is not renormalizable.

The orbifold alternative consists in dividing out the extra dimension by a discrete symmetry which is not freely acting. For illustration we consider only the simplest case of Z_2 parity $y \mapsto -y$.[8] We are interested in the case where the parity does not commute with the gauge group (otherwise all of $\mathcal{A}_5$ will be odd, thus gain Kaluza-Klein (KK) masses of order the compactification scale, and just return us at low energies to a four dimensional world of Yang-Mills with gauge group G). The gauge fields satisfy $\mathcal{A}_M(x,y) = P\mathcal{A}_M(x,-y)P$. (It is to be understood that P also maps any 5th Lorentz component to minus itself.) Providing P generates an automorphism of the Lie algebra of G, such an action is consistent, but the result is that the gauge symmetry is typically restricted to a smaller semi-simple group H at the orbifold points $y = 0, \pi R$. Indeed, we can always choose a basis for the Lie algebra so that the action of P is diagonal. The even generators, $PT^{\hat{a}}P = T^{\hat{a}}$, generate the subgroup H, whilst all the other generators $T^{a'}$ are odd. Although from the low energy four-dimensional point of view, the gauge group G appears to have been broken to H, this is not really true. Gauge transformations $\delta\mathcal{A}_M = \nabla_M\Omega$ are only restricted in the sense that Ω must also satisfy $\Omega(x,y) = P\Omega(x,-y)P$, and thus the components $\Omega^{a'}$ vanish at the boundaries $y = 0, \pi R$.

[8] Most of our comments apply equally well to more involved cases, the most general case in this situation being $Z_2 \times Z_2'$.

As noted in ref. [47], the orbifolding of the 5th dimension results in divergent quantum corrections which are localised at the orbifold points. We are thus forced to introduce brane lagrangians at these points even if we did not start out with them, with couplings of the same form as the divergences (so that renormalizing the couplings absorbs the divergences). By the symmetry at the branes, we need four times as many couplings $1/g_i^2$ as there are semi-simple factors in H, to multiply separately the generated terms $(\mathcal{F}^{\hat{a}}_{\mu\nu})^2$ and $(\mathcal{F}^{\hat{a}}_{\mu 5})^2$ on each boundary (see section 4 for an example). Using the lagrangian in the form (2.2), and for example working in background field gauge [48] cf. figure 1, it is easy to see that the couplings then run according to $\mu\partial_\mu(1/g_i^2) = b_i/(8\pi^2)$ where these b_i are combinations of group theory factors, yielding numbers say, in the range $O(1)$ to $O(10)$, but whose value depends on the details (in particular the orbifolding and the fermions we are about to introduce).

Note that in computing these effects we work in 5 dimensions: the localised divergences only arise from integrating over the four-dimensional loop momentum $\bar{k}^\mu$, whilst the 5th component only participates in the Umklapp process $\sim \delta_{2k_5, p_5 \pm p'_5}$ which in position space leads to the brane localised delta functions $\delta(y) + \delta(y - \pi R)$ [47].[9] We would get the wrong answer if we worked in $4+\epsilon$ dimensions, thus expanding the $\bar{k}$ momentum integral around three dimensions. Indeed in this case the diagrams would all be ultraviolet finite, and all infrared finite, except for the $k_5 = 0$ term which is linearly IR divergent for vanishing p. Whether we work in 5 dimensions or expand around 4 dimensions has to be determined by the effect being calculated: we need to choose to expand about 4, if and only if it is the critical dimension for the effect in question, i.e. the dimension at which logarithmic divergences appear. At one loop, this clearly decides the issue. Perhaps at two loops and higher, there can be an effect that can only be properly understood by considering its divergences both in 5 and around 4 dimensions? We leave this question for the future.

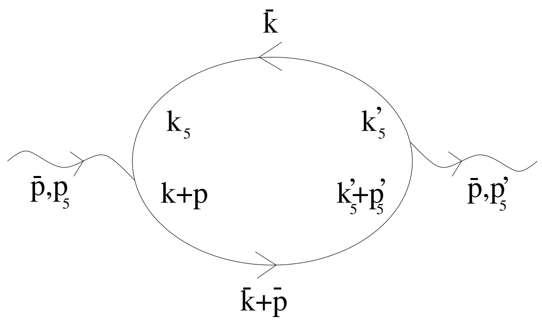

Figure 1: One-loop contribution to the β function in background field gauge. The internal line stands for all components, including ghosts.

Note that it is not possible to argue that the $1/g_i^2$ are much less than $b_i/(8\pi^2)$, because the result would then be strongly μ dependent. Indeed in this case, changing the value of μ by a factor of two, would return the couplings to $1/g_i^2(\mu) \sim b_i/(8\pi^2)$. We thus conclude that the brane couplings are $O(10^{-1})$ or greater.

Even if we were to limit the appearance of these brane kinetic terms by arranging the b_i to be very small or cancel (at one loop or to any number of loops), it is not possible to set $1/g_i^2$ to zero if there are other non-vanishing brane interactions involving $\mathcal{A}$, without generating further divergences, as we explain below.

[9] The loss/gain of KK momentum together with possible reflection of the momenta is analogous to Umklapp interactions with lattices in condensed matter. Indeed in reality the branes absorb the change in momentum.

Since these couplings $1/g_i^2$ do not respect the full gauge symmetry G, Grand Unification in a single gauge group is not really possible in these scenarios (even if some of its features such as charge quantization can be preserved [18]). For this reason, and the fact that lack of renormalizability is less of a problem in any case at Grand Unified scales (cf. section 1), we do not pursue the possibility of renormalizable extra dimensional Grand Unified Theories further here.

To see that it is not possible to switch off brane kinetic terms for $\mathcal{A}$ in the presence of other non-vanishing brane interactions for $\mathcal{A}$, consider adding an H invariant scalar field $\phi(x)$ to the brane(s). This will generate from the kinetic term in particular a brane interaction of the standard "seagull" type

$$\delta(y)\mathcal{A}\mathcal{A}\phi\phi \tag{3.1}$$

to be added to the overall lagrangian density (2.2). The effect we are about to describe happens for any field propagating in the bulk and any interaction, so for the moment we will ignore the Lorentz and colour indices, treating $\mathcal{A}$ as though it were a scalar field. Since this is a small-distance effect, it occurs just as well in infinite flat bulk dimensions with a single brane at $y = 0$.

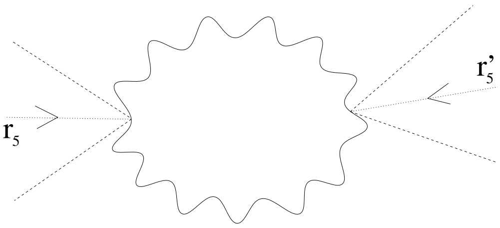

Figure 2: Contribution to brane-localised ϕ^4 interaction. The dashed lines are external ϕs. The dotted lines are insertions of external y-momentum from (3.2).

Now the one-loop diagram indicated in figure 2 generates a local divergent correction to the brane $\phi^4(x)$ interaction. The delta function in (3.1) is easily taken into account if we regard it as a background field

$$\phi_b(x, y) = \delta(y) \tag{3.2}$$

in a five-point interaction, as indicated in figure 2. We see that the local interaction generated is actually

$$\phi_b^2\phi^4 \tag{3.3}$$

which is not well defined with the identification (3.2).

Part of the problem of course lies in our assumption that the brane is infinitely thin and infinitely heavy. In a fully realistic situation, the brane would have a finite mass which we identify with the scale of quantum gravity Λ, and a form factor

$$\phi_b(x, y) = f_b(y)\,, \tag{3.4}$$

where f_b becomes a delta function only in the limit $\Lambda \to \infty$. If $\mathcal{A}$ were not a gauge field, we could indeed replace $\delta(y)$ by (3.4). Since for functions smooth at distances $1/\Lambda$,

$$\phi_b^2 \equiv c\Lambda\phi_b\,, \tag{3.5}$$

(where c is some constant of order 1) this extra divergence could be absorbed into the couplings. However, in our gauge theory, the replacement of $\delta(y)$ by a form factor would be fatal: in order for (3.1) to remain gauge invariant, ϕ has to be extended so that it depends on the bulk dimensions. This would mean we would not have the option to have brane-localised fields, severely limiting not only the model-building possibilities but also the possibility to keep the bulk theory asymptotically free in four dimensions and thus supportable by an ultraviolet fixed point in $4+\epsilon$ dimensions. An alternative way to proceed might be to use (3.2) and promote relations such as (3.5) to identities. At this stage it is not clear if the model can then be made fully renormalizable.

Fortunately we do not have to pursue these directions further, since the divergence does not appear if we keep a brane kinetic term for $\mathcal{A}$. To see this, again for simplicity we treat $\mathcal{A}$ as a single component real scalar field. After a rescaling the kinetic terms yield the inverse propagator

$$\Delta^{-1} = -\,\partial^M \partial_M - \frac{2}{m}\,\delta(y)\,\partial^\mu \partial_\mu\,, \tag{3.6}$$

where $m \equiv 2g_i^2/g^2$ is the ratio of couplings and has dimensions of mass. The corresponding propagator has already been derived in the literature (see e.g. [49]). However, to discuss divergences it is helpful to write it completely in momentum space. This is easily found directly as follows. In euclidean momentum space,

$$\Delta^{-1}(\bar{p}, p_5; \bar{q}, q_5) = p^2 \delta(p-q) + 2\,\frac{\bar{p}^2}{m}\,\delta(\bar{p}-\bar{q})\,, \tag{3.7}$$

where $\delta(k)$ $(\delta(\bar{k}))$ means the $D = 5(4)$ dimensional delta function multiplied by $(2\pi)^D$. Substituting the ansatz,

$$\Delta(\bar{p}, p_5; \bar{q}, q_5) = a(p)\delta(p-q) + b(\bar{p}, p_5, q_5)\delta(\bar{p}-\bar{q})\,, \tag{3.8}$$

where $a(p)$ and $b(\bar{p}, p_5, q_5) = b(\bar{p}, q_5, p_5)$ are functions to be determined one finds

$$\Delta(\bar{p}, p_5; \bar{q}, q_5) = \frac{\delta(p-q)}{p^2} - \frac{2\bar{p}^2\,\delta(\bar{p}-\bar{q})}{p^2(\bar{p}^2+q_5^2)(m+|\bar{p}|)}\,. \tag{3.9}$$

(The modulus term $|\bar{p}| = \sqrt{\bar{p}^2}$ arises from integration over q_5.)

Taking account of momentum conservation in the usual way, one finds the contribution figure 2 is proportional to

$$\int d^4x\, d^4x'\, \phi^2(x,0)\, \phi^2(x',0) \int \frac{d^4\bar{p}}{(2\pi)^4}\, \mathrm{e}^{i\bar{p}.(x-x')} \int \frac{d^4\bar{q}}{(2\pi)^4}\, \Delta(\bar{q})\Delta(\bar{q}+\bar{p})\,, \tag{3.10}$$

where the effective four-dimensional propagator

$$\Delta(\bar{p}) = \frac{1}{\bar{p}^2 + m|\bar{p}|} \tag{3.11}$$

comes from integrating over the injection of r_5 (r_5') with coefficient 1, that arises at each vertex from (3.2).

Note that the effective propagator results in no worse ultraviolet divergences than a four dimensional theory completely confined to the brane. It is clear that all one-loop diagrams of this sort will be regulated in the same way and thus no terms of the form (3.3) are

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generated. Intuitively one can understand what is happening as follows. For very small distances on the brane, the $\delta(y)$ term in (3.6) dominates. In this regime everything is confined to the brane and we have just a four dimensional theory. The five dimensional coupling constant g, through m, provides a cross-over scale in the effective propagator (3.11), so that only for distances larger than $1/m$ do we feel the effects of the full 5 dimensions. In fact what we have is the gauge theory analogue of the "DGP" model [49], where these effects are discussed in a gravitational context, a neat derivation of precisely (3.11) appearing in ref. [50]. This intuition gives us confidence that this ultraviolet regularisation effect works to all loop orders.

Finally, note that the above considerations imply that all the β functions of the brane-localised fields (here that of a brane scalar field) must be singular in the limit $1/g_i^2 \to 0$, in which the brane kinetic terms are turned off.

Note that the gauge field components $\mathcal{A}_5^{a'}$ are even under P. Their lowest KK modes thus behave like scalars as far as the low energy four-dimensional theory is concerned. There is nothing in the four-dimensional theory that prevents them from gaining a mass. From the five dimensional point of view this is not allowed by gauge invariance. However, an effective action can and will appear for the non-local operator

$$\Phi_P(x) = \Phi(x, 2\pi R), \tag{3.12}$$

the Wilson line that winds around the compact dimension, where

$$\Phi(x, y) = \mathcal{P} \exp -i \int_0^y d\tilde{y}\, \mathcal{A}_5(x, \tilde{y}) \tag{3.13}$$

is the Wilson line from $\tilde{y} = 0$ to $\tilde{y} = y$, located at x (and $\mathcal{P}$ stands for path ordering). Since under gauge transformations,

$$\delta\Phi(x, y) = i\,\Omega(x, 0)\,\Phi(x, y) - i\,\Phi(x, y)\,\Omega(x, y), \tag{3.14}$$

$\Phi_P(x)$ transforms homogeneously as an adjoint scalar at $(x, 0)$.

Since Φ_P contains a power series in $\mathcal{A}_5^{a'}$, the gauge invariant effective potential $\mathrm{tr}\, V_{\text{eff}}(\Phi_P)$ not only generically results in masses for the lowest KK mode of the components $\mathcal{A}_5^{a'}$, but can also result in spontaneous symmetry breaking: $\langle \mathcal{A}_5^{a'} \rangle \neq 0$. This is the Hosotani mechanism [34]. In the spontaneously broken phase, although we can by a gauge transformation set $\mathcal{A}_5 = 0$, this is at the expense of a non-trivial Scherk-Schwarz twist in periodicity conditions for $\mathcal{A}$ [51] (as we review in section 4) which also breaks the gauge group. Meanwhile, V_{eff} and the low energy physics as described through it, are completely unaffected by this gauge transformation. Note that the mass and more generally the effective potential is protected from divergent corrections: the effect is non-local in the full five-dimensional space and thus quantum corrections are cutoff by the compactification scale $1/(2\pi R)$.

Clearly an extremely attractive possibility now arises, namely to regard the Higgs scalar field as the components $\mathcal{A}_5^{a'}$ [34]–[36],[6]–[13], for if we can bring such a model into accord with phenomenological constraints, whilst keeping it renormalizable, we will have

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solved the hierarchy problem. This is the possibility we will consider in more detail in section 4. We will use the framework developed there to make general comments on model building constraints that arise when we consider couplings to fermions.

4. The simplest gauge-Higgs unification model

The simplest possibility for the unified gauge group is $G = \mathrm{SU}(3)$ [52].[10] With the generators $T^a = \lambda^a/2$, where the λ^a are the Gell-Mann matrices, we can decompose $\mathcal{A}_M$ as

$$\mathcal{A}_M = \begin{pmatrix} W_M & H_M/\sqrt{2} \\ H_M^\dagger/\sqrt{2} & 0 \end{pmatrix} + B_M T^8 \,. \tag{4.1}$$

Here, $(W_M)^i_j$, with $i,j = 1,2$, contracted into $T^{\hat{a}}$ ($\hat{a} = 1,2,3$), are the gauge bosons associated to the top left SU(2) subgroup. This will be identified with the $\mathrm{SU}(2)_W$ part of the Standard Model gauge group. Similarly B_μ will be identified with the $\mathrm{U}(1)_Y$ gauge boson, while $H^i \equiv H^i_5 = \sqrt{2}(\mathcal{A}_5)^i_3$ is to be identified with the $\mathrm{SU}(2)_W$ doublet Higgs. We implement the restriction of SU(3) to $\mathrm{SU}(2)_W \times \mathrm{U}(1)_Y$, by identifying the parity operator P as mapping the third component of the SU(3) fundamental representation to minus itself, whilst leaving components $i = 1,2$ alone. Under this map indeed, W_μ, B_μ and H are even and thus at this stage will have massless KK modes, whilst W_5, B_5 and H_μ are odd and thus only have KK masses of order the compactification scale or greater.

The radiatively generated potential for Φ_P, will then spontaneously break the model, effectively giving H a vacuum expectation value, via the Hosotani mechanism as outlined in the previous section. Using the remaining global symmetry we can ensure the vacuum expectation value takes the form $\langle H \rangle = (0, v)/\sqrt{2}$ for some real v. Thus $\mathrm{SU}(2)_W \times \mathrm{U}(1)_Y$ will be broken down to electromagnetism, just as happens in the Standard Model.

Since

$$[T^8, H] = \frac{\sqrt{3}}{2} H \,, \tag{4.2}$$

we already have a problem however. If we were to put the coupling g back in its normal place in the covariant derivative, we would want to identify gT^8 with $g'Y/2$ where Y is the hypercharge and g' the associated coupling. Comparing to the Standard Model Higgs for which $Y = 1$, we get from (4.2), that $g' = \sqrt{3}g$ and thus we 'predict' $\sin^2\theta_W = g'^2/(g^2 + g'^2) = 3/4$, a phenomenological disaster.

However, we now recall that we necessarily have the following couplings on the branes:

$$\Delta\mathcal{L} = \delta(y)\,\mathcal{L}_1 + \delta(y - \pi R)\,\mathcal{L}_2 \,, \tag{4.3}$$

where, writing out the non-vanishing parts of $\mathcal{F}^2_{\mu 5}$ and similarly the semi-simple components of $\mathcal{F}^2_{\mu\nu}$,

$$\mathcal{L}_\alpha = -\frac{1}{2w_\alpha^2}\,\mathrm{tr}\, W^2_{\mu\nu} - \frac{1}{4b_\alpha^2}\, B^2_{\mu\nu} + \frac{1}{h_\alpha^2}\,|\nabla_\mu H - H'_\mu|^2 \,. \tag{4.4}$$

[10] This is really SU(3) × SU(3). The second SU(3) is for strong interactions and is readily incorporated in the bulk. Since the real problems lie with the weak interactions we will not discuss colour further.

Here we have defined $H'_\mu = \partial_5 H_\mu$, the brane couplings w_α, b_α and h_α, and the field strengths $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ and $W_{\mu\nu} = i[D_\mu, D_\nu]$, where $D_\mu = \partial_\mu - iW_\mu$. Note that

$$\nabla_\mu H = \partial_\mu H - iW_\mu H - i\frac{\sqrt{3}}{2} B_\mu H\,, \tag{4.5}$$

as a consequence of (4.2) and the restriction to the branes.

Recall that the couplings $1/w^2$, $1/b^2$ and $1/h^2$ are $O(1/10)$ or larger. (The symmetric point where they have the same value on each brane will typically be broken when fermions are included.) In the presence of such couplings (4.4), spontaneous symmetry breaking deforms the lowest mass modes for the weak vector bosons $W^\pm_\mu$ and Z_μ so that they are no longer simply constants in the y direction (in any gauge) [13].

However, this effect is controlled by the ratio of scales $\theta/2\pi = Rv/2$ [13], vanishing as $\theta \to 0$. This ratio appears in

$$\langle \Phi_P \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -i\sin\theta \\ 0 & -i\sin\theta & \cos\theta \end{pmatrix}. \tag{4.6}$$

The physics is invariant under $\theta \mapsto \theta + 2\pi$. Defining θ to be in the fundamental domain $[0, 2\pi)$, the natural theoretical expectation is that $\theta/2\pi \sim 0.5$ if spontaneous symmetry breaking takes place. However, indirect limits require the compactification scale $1/R = a\,\mathrm{TeV}$ where $a \gtrsim 2-5$ or greater [53]. Identifying v for the moment with the Standard Model Higgs' vacuum expectation value, we require $\theta/2\pi \lesssim 0.06$. In this case we can ignore the deformations in the first approximation and take the lowest mass modes for $W^\pm_\mu$, Z_μ and H to be constant in the y direction.

The form of the effective lagrangian for these modes is then

$$\begin{aligned} &2\pi R\left[-\frac{1}{2g^2}\,\mathrm{tr}\;W^2_{\mu\nu} - \frac{1}{4g^2}B^2_{\mu\nu} + \frac{1}{g^2}|\nabla_\mu H|^2\right] - \\ &-\frac{1}{2}\,\mathrm{tr}\;W^2_{\mu\nu}\sum_\alpha \frac{1}{w^2_\alpha} - \frac{1}{4}B^2_{\mu\nu}\sum_\alpha\frac{1}{b^2_\alpha} + |\nabla_\mu H|^2 \sum_\alpha \frac{1}{h^2_\alpha}\,, \end{aligned} \tag{4.7}$$

where (4.5) holds but now with y-constant modes. Thus clearly if we define

$$\frac{1}{g_2^2} = \frac{1}{2\pi^2\tilde\alpha} + \frac{1}{w_1^2} + \frac{1}{w_2^2} \tag{4.8}$$

$$\frac{3}{g_1^2} = \frac{1}{2\pi^2\tilde\alpha} + \frac{1}{b_1^2} + \frac{1}{b_2^2} \tag{4.9}$$

where we have used the Wilsonian dimensionless bulk coupling (2.5) evaluated at $\mu = 1/R$ (and $D = 5$, $N = 3$), viz. $\tilde\alpha = g^2/4\pi^3 R$, and redefine

$$W_\mu = g_2\,\tilde W_\mu\,, \qquad B_\mu = \frac{g_1}{\sqrt{3}}\,\tilde B_\mu\,, \qquad H = \tilde H\left(\frac{1}{2\pi^2\tilde\alpha} + \frac{1}{h_1^2} + \frac{1}{h_2^2}\right)^{-1/2}, \tag{4.10}$$

to put couplings back in their usual places, we get back precisely the relevant part of the Standard Model lagrangian

$$-\frac{1}{2}\,\mathrm{tr}\,\tilde W^2_{\mu\nu} - \frac{1}{4}\tilde B^2_{\mu\nu} + |\nabla_\mu \tilde H|^2 \tag{4.11}$$

where in particular $\tilde{W}_{\mu\nu}$ is now defined as expected in terms of $D_\mu = \partial_\mu - ig_2\tilde{W}_\mu$, and the full covariant derivative

$$\nabla_\mu = \partial_\mu - ig_2\tilde{W}_\mu - ig_1\frac{Y}{2}\tilde{B}_\mu \tag{4.12}$$

defines a hypercharge

$$Y = \frac{\mathrm{diag}(1,1,-2)}{3} \tag{4.13}$$

which gives the Higgs $Y = 1$ as expected.

Taking into account the rescaling to $\tilde{H}$, we see that v is actually defined in terms of the Standard Model Higgs' expectation value $\tilde{v}$ by

$$R\,\tilde{v} = \frac{\theta}{\pi}\sqrt{\frac{1}{2\pi^2\tilde{\alpha}} + \frac{1}{h_1^2} + \frac{1}{h_2^2}}\,. \tag{4.14}$$

These equations allow us to make several straightforward but important conclusions. Firstly, in the limit that θ is sufficiently small to neglect deformations of the lowest KK wavefunctions, we regain from (4.11), the custodial symmetry of the Standard Model and thus determine the ρ parameter to be one at tree level, as required. Secondly, using the experimentally determined numbers [54], the bound $1/R = a\,\mathrm{TeV}$ and the fact that $1/w^2$, $1/b^2$ and $1/h^2$ are $\gtrsim 1/10$ we can place further bounds on the values of the parameters.

Thus from (4.8), $1/2\pi^2\tilde{\alpha} < 2.3$, and with natural values for w_α, we have $\tilde{\alpha} \gtrsim 0.02$. This is clearly perturbative and an order of magnitude smaller than the typical fixed point value (cf. section 2). At energies higher than $1/R$, $\tilde{\alpha}$ will run according to (2.6), reaching the non-perturbative physics associated with the fixed point at energy scales $\sim 10/R$.

On the other hand we clearly have an upper bound on $1/w_1^2 + 1/w_2^2$ of 2.3. From (4.9) the strict upper bound on $1/2\pi^2\tilde{\alpha}$ implies a *lower* bound on $1/b_1^2 + 1/b_2^2 > 21$. We therefore require very large $\mathrm{U}(1)_Y$ kinetic terms on the boundary. Perhaps these arise naturally from non-perturbative physics close to the ultraviolet fixed point $\tilde{\alpha}_*$, bearing in mind that abelian gauge fields would be separately non-renormalizable in the bulk. Of course phenomenologically, the origin of the large values is the discrepancy between the 'bulk' $\sin^2\theta_W = 3/4$ and the experimental one.

The lagrangian given by (2.2) and (4.4) is very similar to the bosonic sector of the lagrangian considered in ref. [13], except that Scrucca et al. introduce an extra bulk U(1) gauge field to allow $\sin^2\theta_W$ to be set to its experimental value (and do not introduce the h_α terms). We do not have the option of including a bulk U(1) gauge field since it is not renormalizable within the ϵ expansion.

Finally, (4.14) implies that $\theta/2\pi > 0.08/a$ with natural values for $1/h_\alpha^2$. Therefore phenomenologically preferred values of θ are consistent with our approximation.

To allow for larger θ, we have to increase $\tilde{\alpha}$. Thus $\tilde{\alpha} \sim 0.1$ (implying $1/w^2 \sim 1$) allows for $\theta/2\pi \sim 0.18/a$, but to get the theoretically natural values of $\theta/2\pi \sim 0.3$ [13] requires $\tilde{\alpha} \sim 1.2$. Such a large value would probably (depending on matter content) imply that the fixed point $\tilde{\alpha}$ is being approached from the right (i.e. from the region $\tilde{\alpha} > \tilde{\alpha}_*$). There is no problem of principle with this happening but in particular, large higher order corrections in the ϵ expansion mean that we could then no longer trust these simple formulae. However, in any case we are back in a regime where significant distortions from Standard Model relations will be found. We should note that there are not enough parameters in the model in this regime to tune away the resulting anomalous triple gauge couplings [13, 55], tune ρ, $\sin^2\theta_W$ and m_Z and the effective brane couplings to matter fields [13] to their correct values. Even if we could manage this, it would be an unsatisfactory accident since custodial symmetry has been badly broken.

In summary, for the bosonic sector of the model to be phenomenologically acceptable we must have large values of $1/b^2 \sim 10$, and a much smaller θ than we would find without some special mechanism. Indeed we can choose a small $\tilde{\alpha} > 0.02$ and natural values of $1/w^2 \sim 1/h^2 \sim 1/10$, in which case we have to arrange the model to dynamically determine $\theta/2\pi \sim 0.04 - 0.09$.

Before turning to the introduction of fermions we discuss briefly the peculiar remnants of gauge invariance operating on the brane. Decomposing a gauge transformation similarly to (4.1), as

$$\Omega = \begin{pmatrix} \omega & \phi \\ \phi^\dagger & 0 \end{pmatrix} + \beta T^8 \,, \tag{4.15}$$

we have that ω and β are P even, while ϕ is P odd. Thus ω, β and $\phi' = \partial_5\phi$ survive on the branes whilst $\partial_5\omega$, $\partial_5\beta$ and ϕ vanish on each brane. It follows that the brane lagrangians (4.4) are invariant under the surviving, or "remnant" [11], gauge symmetries:

$$\begin{aligned} \delta W_\mu &= [D_\mu, \omega] \\ \delta B_\mu &= \partial_\mu \beta \\ \delta H &= \phi' + i\omega H + i\frac{\sqrt{3}}{2}\beta H \\ \delta H'_\mu &= i\omega H'_\mu + i\frac{\sqrt{3}}{2}\beta H'_\mu + \nabla_\mu \phi' \,, \end{aligned} \tag{4.16}$$

where $\nabla_\mu \phi'$ is (4.5) with H replaced by ϕ'.

As noted in ref. [11], the shift symmetry ϕ' protects against brane mass terms appearing for the Higgs. It is interesting also to note that if it were not for the bulk lagrangian (2.2), the brane Higgs kinetic term in (4.4) would be trivial since it could be gauged away by a finite $\phi' = H$ transformation in (4.16), H'_μ playing the rôle of an auxiliary field. However, this term becomes non-trivial when considered as part of the full lagrangian: although we can still gauge the brane Higgs kinetic terms away, this is a gauge choice which in general conflicts with the need to make other gauge choices (for example the background Feynman gauge used typically to compute the radiative potential). Furthermore, H'_μ is of course no longer an auxiliary field but part of the bulk degrees of freedom evaluated at the orbifold points.

In order to complete a realistic description we need to introduce quarks, and leptons. For example consider initially, the top and bottom quarks. We write their $\mathrm{SU}(2)_w \times \mathrm{U}(1)_Y$ $\mathbf{2}_{1/6}$, $\mathbf{1}_{2/3}$, $\mathbf{1}_{-1/3}$ representations as $Q_L = (t_L, b_L)$, t_R and b_R. As usual in gauge-Higgs unification models, we cannot introduce these as bulk fermions because the interactions would be flavour-symmetric and have the wrong hypercharges, following from (4.13). If we introduce them as fields that live only in the brane(s), then we can simply assign them the correct hypercharges as according to (4.12), since only the restricted symmetry (4.16) is active there. However, (4.16) as well as including the standard $\mathrm{SU}(2)_W \times \mathrm{U}(1)_Y$ gauge transformations (after use of (4.10) and defining $\tilde{\beta} = \sqrt{3}\beta$) includes the shift symmetry generated by ϕ' which forbids the Yukawa interactions, for example under the shift symmetry we have $\delta(\bar{Q}_L H b_R) = \bar{Q}_L \phi' b_R$.

(We can cancel this by postulating heavy mirror fields, e.g. a $\mathbf{2}_{1/6}$ Q_R, adding appropriate mass terms $M\bar{Q}_L Q_R$ (plus c.c.), and defining $\delta Q_R \sim \phi' b_R$, and so on, resulting in a see-saw mechanism with the lightest states to be identified with the quarks. However we then need kinetic terms for the Q_R. The terms generated by the shift transformations acting on this can be cancelled by introducing an appropriate interaction with H'_μ and appropriate transformations into heavy partners for b_R and t_R. Unfortunately then more fields are needed to in order to cancel new violations of the shift symmetry and so on. It appears that it is not possible to find a closure of the symmetry which is non-trivial, linear and finite dimensional. However, since we are already introducing an infinite number of fields through the KK excitations, it could be worthwhile to pursue the possibility of infinite dimensional representations of the shift symmetry. Note that if these could be constructed, the shift symmetry would still protect the Higgs from gaining a brane potential, but allow standard Yukawa interactions for the real quarks and leptons.)

This problem has been circumvented in the literature by taking instead the Wilson line Φ, using Φ_P for coupling doublets and singlets on the brane at $y = 0$ (analogously a Wilson line that wraps once round the compact dimension but starting at $y = \pi R$, for coupling both representations at $y = \pi R$), or $\Phi(\pi R)$ for coupling fermions on one brane to fermions on the other [9]. This can be done because from (3.14),

$$\delta\, \Phi^i{}_3(x,y) = i\omega(x,0)\Phi^i{}_3(x,y) + \left[\frac{i}{6}\tilde{\beta}(x,0) + \frac{i}{3}\tilde{\beta}(x,y)\right] \Phi^i{}_3(x,y) \qquad \text{for } y = 0, \pi R\,. \tag{4.17}$$

Therefore $\Phi^i{}_3(x,y)$ transforms homogeneously like the Higgs in (4.16) but without the shift symmetry. From the above we see we can actually provide Yukawa couplings between different branes as $\bar{Q}_{Li}(x,0)\Phi^i{}_3(x,\pi R) b_R(x,\pi R)$, but $\mathrm{U}(1)_Y$ invariance requires the standard coupling for the charge conjugates Q^c_R and t^c_L to be on the same brane. We may similarly provide couplings for the other quark families, and also for the leptons, where we just use the fact that for Φ_P, (4.17) with $y = 0$ is the usual Higgs transformation.

Such Wilson line interactions can arise from integrating out heavy bulk fields [9, 13, 8] and also arise in String compactifications [56, 9]. There is no problem here and later with brane localised anomalies since they may be cancelled by an appropriate bulk Chern-Simons action [57]. However as noted by ref. [9] such Yukawa interactions produce new brane-localised divergences. In particular they will give quadratically divergent contributions to the Higgs mass

$$\sim \Lambda^2 \Phi^i{}_3 \Phi^{\dagger 3}{}_i \tag{4.18}$$

just as in the Standard Model. Of course this destroys the purpose of the model as a solution to the hierarchy problem.

The reason is immediately clear because the y degree of freedom plays no rôle here for the brane localised fermions. As far as they are concerned $\Phi^i{}_3$ looks just like the

Standard Model Higgs. Indeed we would also have to add brane kinetic terms and a brane-localised quartic potential for ${\Phi^i}_3$ to absorb logarithmic divergences and make the model renormalizable.[11]

In ref. [9], the authors are interested only in the little hierarchy problem, and circumvent the further difficulty (4.18) for the top by adding a new colour triplet fermion (for us this would be a $\mathbf{1}_{-1/3}$) so as to complete the representation Q_L to a representation of $\mathrm{SU}(3)_W$. Then the summation over i in (4.18) in fact runs over the complete representation ($i = 1, 2, 3$) becoming just a contribution to the vacuum energy ($\Phi\Phi^\dagger = 1$). The divergences from the other fermions are small enough to ignore in this scenario since the model anyway has a low effective cutoff Λ and the corresponding Yukawa couplings are much smaller. With a Planck mass cutoff, we do not have this option, even for neutrinos and their very low masses, and clearly it is phenomenologically unacceptable to add hypercharged but $\mathrm{SU}(2)_W$ singlets to fill out representations of $\mathrm{SU}(3)_W$ for the other quark families and the leptons.

The appearance of divergences from such non-local operators seems to go against the standard lore that divergences arise from local interactions only. There is no contradiction however. These Wilson line interactions are local in space-time (x space). This is an approximation which is valid only in the limit of infinite string tension (in the case of String Theory compactifications) or infinite mass (in the case of massive bulk fields). In truth one should integrate over different locations x and x' where the ends of the Wilson line meet the fermions, with the displacement $x - x'$ being weighted by a form factor of width $1/M$, where M is the heavy scale. These Wilson line interactions are then indeed ultraviolet finite, quantum corrections being naturally cutoff at M.

Since our model is supposed to be valid up to the Planck mass Λ however, we are forced to describe the dynamics that results in such smeared Wilson line interactions (unless $M \gtrsim \Lambda$ in which case we are back at square one). We are left with the remaining concrete alternative in the literature which is indeed to generate the Yukawa interactions through couplings of bulk and boundary fields [13, 10].

We follow closely the ideas of ref. [13]. Thus we introduce a pair of bulk fermions $\Psi_1(x, y)$ and $\Psi_2(x, y)$ (necessarily Dirac fermions since they are in 5 dimensions) with opposite y-parity so that a parity invariant mass term $M\bar{\Psi}_1\Psi_2$ (plus c.c.) can be built. Using (4.13), we choose a set of representations of SU(3) so that the Standard Model fermions, which exist only on the branes, can all couple to them and thus all the Standard Model fermions (including the neutrinos) can get masses from the effective Wilson line interactions that will be generated. Thus we can use a Ψ_α pair in the fundamental (**3**) representation so that Ψ^i couples to Q_L while Ψ^3 couples to b_R. This is the minimal representation that will do the job. The minimal representation that yields components with the right hypercharge to couple to t_R is the **6**, where we use the 33 component to couple to t^c_L. (The **6** will also couple to Q^c_R.) We deal with the other quark families in precisely the same way. Charged conjugated lepton doublets can couple first to the adjoint

[11] It is not clear to us whether the resulting framework is renormalizable in 5 dimensions, however if it is not, then the right approach is to consider these effects expanded around 4 dimensions, where it is renormalizable — see our comments below (3.11).

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(**8**) representation; the charge conjugated right handed electron, muon and tau couple to the 333 component of a **10**, and finally the right handed neutrinos can couple to the $\mathbf{1}_0$ in the **8** representation Ψ_αs already introduced.

As we have already discussed, something special is needed to get a low value $\theta/2\pi \sim 0.04 - 0.09$. In ref. [58] it was shown that this is possible by adding of order 10 bulk fields in a mixture of fundamental and adjoint representations. They used scalars as well as fermions. Although scalars are problematic for us, they do not appear to be especially required. These authors assume that the correct value of $\sin^2\theta_W$ can be realised by wall localised kinetic terms. As we have shown, this can indeed be achieved. Somewhat similarly, the authors of ref. [13] manage to reduce the minimum to $\theta/2\pi \sim 0.096$ by adding bulk fermions in large (rank 8) symmetric representations. They note however that this will lead to electroweak corrections enhanced by large group theoretical factors resulting in the scale at which the bulk weak coupling becomes non-perturbative being lowered (their cutoff by application of NDA). Here we do not have this problem.

For quite separate reasons we have also introduced a large number of bulk fermions. It would be very interesting to see if the menagerie of representations we have had to introduce to induce Yukawa matrices for the Standard Model fermions, also turned out to give values of θ in the right range. However, we now hit a severe problem if we want to preserve the renormalizability of the model. The contributions of the bulk fermions to the bulk β function is[12]

$$3\beta_0 = -8\sum_R T_R = -4\,(1+5+6+15) = -108 \tag{4.19}$$

completely overwhelming the contribution of $+11$ from $\mathcal{A}_M$. Thus the bulk theory is no longer asymptotically free in four dimensions and cannot be supported by an ultraviolet fixed point in $4+\epsilon$ dimensions.

This constraint would appear to rule out renormalizable models based on a bulk SU(3) Yang-Mills theory. The obvious route to try to make further progress would be to consider larger gauge groups, thus increasing the gauge-field contribution to β_0, while also allowing more Standard Model fermion representations to couple to the same bulk fermion representations. (For example above, the **3** is shared by Q_L and b_R, while the right handed neutrinos couple to the same representation as the lepton doublets.) This results in considering a kind of grand unification of a quite different sort from the standard four dimensional cases, but with some similar properties, for example charge quantization and mass relations. However, we also clearly need to consider more complex orbifolds, and/or deal with multiple Higgs vacuum expectation values arising from the Hosotani mechanism (see ref. [59] for such a study).

5. Summary and conclusions

Extra dimensional field theories have the potential to solve many of the enduring mysteries of theoretical particle physics. However approaches that directly address weak-scale physics

[12]The factor 8 comes from the usual 4/3, the two bulk fermions and the three families. Note that boundary fields and couplings make no contribution to the bulk β function.

and in particular the hierarchy problem, suffer from a severe drawback in that they are not renormalizable, at least as conventionally envisaged. This results in an irreducible uncertainty of typically $\sim 1\%$ in any predictions following from these models, and also implies the existence of a scale much smaller than the Planck mass ($\sim 100\,\mathrm{TeV}$) where something other than field theory has to take over.

It seems possible that a restricted class of such theories may however be made renormalizable by basing their continuum limit around a non-perturbative fixed point (rather than the perturbative gaussian fixed point that supports the Standard Model).

Although it has been recognized since the 1970s [23] that non-abelian Yang-Mills theory might have such extra dimensional ultraviolet fixed points, only a few studies have been made to search for these. From the lattice studies it seems clear that the simplest Wilson plaquette bare action does not allow these fixed points for SU(2) Yang-Mills in $D = 5$ or 6 dimensions. However, as we emphasised, there is no reason to expect the simplest action to be the correct bare action in this case.[13] The lattice study in ref. [27] suggests that even with some more general actions the fixed points do not exist in $D = 6$ dimensions and large N. On the other hand, the exact renormalization group study by Gies [24] suggests that these fixed points do exist in $D = 5$ dimensions for SU(N) Yang-Mills at least for $N \leq 5$.

Somewhat surprisingly, no-one seems to have carried the initial Wilson epsilon expansion investigation beyond the early two-loop computation of Peskin [23], so in this paper we do that by extending the investigation to the four loops now available. We find that the ϵ expansions are very well behaved asymptotic series, for example we predict that the coefficients in the large-N limit of the expansion for the fixed point coupling $\tilde{\alpha}_*$ do not start to diverge until ~ 10 loops. We give values for both the fixed point coupling and the critical exponent ν, together with estimates of the error, by following — where justified — the simplest methods possible. In broad terms, it seems clear that the $D = 5$ dimensional fixed points do exist, for all $N \geq 2$. The evidence for fixed points in $D = 6$ dimensions, is marginal, while we find strong evidence that the fixed points do not exist in any dimension $D \geq 7$. As we sought to emphasise, on the one hand ϵ expansions above the critical dimension such as this, are justified from studies in other models, and on the other hand there is considerable room for improvement on the present study by using more sophisticated methods.

More generally, within an expansion in ϵ of the $D = 4 + \epsilon$ dimensional theory, these ultraviolet fixed points exist if and only if the four dimensional theory has only asymptotically free couplings. By using a dimensional regularisation and a minimal subtraction type scheme, we can renormalize the theory perturbatively in the normal way in four dimensions. Where we need to investigate renormalization group properties in the higher dimensions, we can use equations such as (2.6) to analytically continue the results to $\epsilon > 0$. These observations open the door to constructing renormalizable extra dimensional models.

In section 3, we extended this generalisation of renormalizability to include matter and branes. We can add fermions to the bulk providing there are not too many to destroy

[13] Actions with more derivatives than (2.2) would generically have problems with locality at energies of the overall cutoff, but this cutoff can be taken to infinity.

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the $D = 4$ dimensional asymptotic freedom. It is possible to include bulk scalar fields but only for careful choices of couplings. The inclusion of both bulk fermions and scalars is only natural in supersymmetric theories (where such fixed points have been independently discovered [32]). Abelian gauge fields cannot however be added to the bulk.

Several ideas in the literature such as Higgsless theories, or the inclusion of Wilson lines directly in the bare action to generate Yukawa terms, do not extend to renormalizable theories in this way, at least without further development (see sections 3 and 4).

When including branes, care has to be taken to work in the correct critical dimension for the divergences being studied. Thus in the case of $D = 5$ dimensions compactified on an orbifold S^1/Z_2, brane kinetic terms are required from renormalization in $D = 5$ dimensions [47] (and not $D = 4$ analytically continued to 5). Furthermore, we showed that even if these bulk-generated divergences were eliminated in a particular model, brane-bulk interactions would lead to a non-renormalizable theory if the brane kinetic terms were set to zero. The fact that these are avoided for non-zero brane kinetic terms is a gauge theory analogue of the DGP effect [49], and incidentally implies that the beta functions of the brane-localised fields must diverge in the limit that the brane kinetic terms are turned off. These observations have relevance for all such extra dimensional models, not only the renormalizable ones being proposed here.

We further pursue the phenomenological and theoretical constraints placed on renormalizable extra dimensional models in section 4. We concentrate on the weak interactions and gauge-Higgs unification via the Hosotani mechanism [34]: if such models can be made renormalizable, they become solutions to the hierarchy problem. We focus only on the simplest model of gauge-Higgs unification based on $D = 5$ dimensional SU(3) Yang-Mills theory compactified on S^1/Z_2. For the model to be phenomenologically acceptable, and perturbatively renormalizable in the manner we have described, we must have large values of $1/b^2 \sim 10$ for the $\mathrm{U}(1)_Y$ brane kinetic terms, and a much smaller Hosotani vacuum angle θ than we would find without some special mechanism. Indeed we can choose a small $\tilde{\alpha} > 0.02$ (implying that the effects of the fixed point will be felt only at energies 10 times higher than the compactification scale, i.e. at energies $\sim 20 - 50\,\mathrm{TeV}$) and choose natural values for the other brane kinetic terms, in which case we have to arrange the model to dynamically determine $\theta/2\pi \sim 0.04 - 0.09$. Such values would however ensure that at low energies the distortions of the Standard Model are sufficiently small not to come into conflict with precision measurements.

The known fermions must live only on the brane and do not make a contribution to the bulk β functions (at least at one loop). The real problem arises when we consider how to generate effective Yukawa couplings. We seem to be forced to add bulk fermions in representations and number that are too large to maintain asymptotic freedom in $D = 4$ dimensions, thus destroying the non-trivial ultraviolet fixed point in $4 + \epsilon$ dimensions. Note that there is no direct relation between this consequence and the fact that in four dimensions one cannot maintain asymptotic freedom and add sufficiently many scalars to spontaneously break all directions in the gauge group [44]: the group in higher dimensions can be much larger and broken in the first place by the orbifold boundary conditions.

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Indeed, in order to make progress, one should consider a larger group and more involved compactifications. As we noted in section 4, the Standard Model fermions can be encouraged to share their interactions with the bulk fermions, again reducing the problem, and leading to a new type of unification, the possibilities and consequences of which deserve further exploration.

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